

A FORMULA IN THE THEORY OF FINITE TYPE INVARIANTS

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ABSTRACT. A family C_{k+1} of local moves on knot diagrams for each positive integer k , is defined in [2] where it is also shown that two knots are C_{k+1} -equivalent iff all of their Vassiliev-Gusarov invariants of degree k agree. Every move C_{k+1} splits the space \mathcal{K} of knots into equivalence classes and defines a metric on each equivalence class. For two C_{k+1} -equivalent knots K, J with $d_{C_{k+1}}(K, J) = 1$ we give a formula for the difference $v_{k+1}(K) - v_{k+1}(J)$. From this we deduce a formula for the difference of the degree $k + 1$ invariants of two knots all of whose degree k Vassiliev invariants coincide.

1. Introduction

We present a formula which expresses the difference of the degree $k + 1$ Vassiliev invariants of two knots K and J which differ by one application of a certain local move¹. In order to be able to state the result we need some terminology which we now explain.

1.1. Tangles, braids and symmetries. Consider a quadrangle with n points on its floor-side labeled p_1, \dots, p_n and n points on its ceiling-side labeled p^1, \dots, p^n as in the figure below:

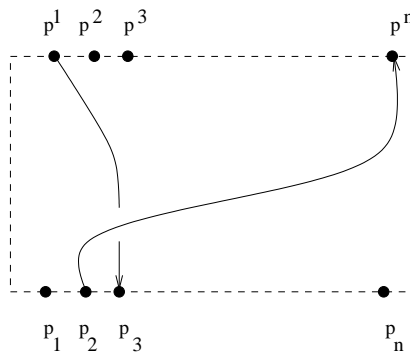


FIGURE 1. The boundary points of braids in the n th braid group with an o -orientation and some connecting oriented strings.

Key words and phrases. knots, unknotting numbers, finite type invariants.

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¹This paper grew out of part of a talk I gave in April of 1999 at the Topology Seminar of the Max-Planck-Institut für Mathematik in Bonn.

These points will be the boundary points of oriented tangles after we introduce one more piece of data. Let $I_n = \{1, \dots, n\}$ and let $o: I_n \rightarrow \mathbb{Z}_2$ be any function which orients the points p_i, p^j as follows: If $o(i) = 0$ then p_i is an incoming point and p^i is an outgoing point while if $o(i) = 1$ then p^i is an incoming point and p_i is an outgoing point (in the figure above $o(2) = 0 = o(n)$ and $o(1) = o(3) = 1$). Considering oriented tangles whose strings connect a p_i to a p^j , if $o(i) = 0 = o(j)$ or a p^j to a p_i , if $o(i) = 1 = o(j)$, gives rise to the n th o -braid group \mathcal{B}_n^o whose end points are oriented according to o .

\mathcal{B}_n^o can be thought of as follows. Let $\Phi_n: \mathcal{B}_n \rightarrow S_n$ be the well known group homomorphism which sends a braid to the permutation it induces on the boundary points. If we use o to impose a partition on I_n and then take the subgroup S_n^o of S_n which respects this partition then \mathcal{B}_n^o can be thought of as $\Phi_n^{-1}(S_n^o)$.

Next consider general tangles with the same boundary points as the ones in \mathcal{B}_n^o (so as to for example allow connecting p_i to p_j when $o(i) + o(j) = 1 \in \mathbb{Z}_2$). Denote the set of all such tangles then by \mathcal{T}_n^o so as to have $\mathcal{B}_n^o \subset \mathcal{T}_n^o$.

As a generalization of the group homomorphism $\Phi_n: \mathcal{B}_n \rightarrow S_n$ there is a map $\Phi_n^o: \mathcal{T}_n^o \rightarrow S_n$ defined by sending a tangle $t \in \mathcal{T}_n^o$ to a bijection $\Phi_n^o(t): \{p_i: o(i) = 0\} \cup \{p^i: o(i) = 1\} \rightarrow \{p_i: o(i) = 1\} \cup \{p^i: o(i) = 0\}$ which sends $x \in \{p_i: o(i) = 0\} \cup \{p^i: o(i) = 1\}$ to $y \in \{p_i: o(i) = 1\} \cup \{p^i: o(i) = 0\}$ if t contains an oriented string connecting x to y (resp. y to x) if $o(x) = 0 = o(y)$ (resp. $o(x) = 1 = o(y)$). Another way to say this is that oriented strings define a bijection from incoming to outgoing points. To pass from this bijection to an element of S_n we need to enumerate the points of the domain and range of Φ_n^o . We do this from left to right. The restriction then of this map Φ_n^o to \mathcal{B}_n^o gives rise to a group homomorphism.

By closure of elements of \mathcal{T}_n^o we will mean the identification of p_i with p^i for each i as in the figure below. For $t \in \mathcal{T}_n^o$ its closure will be denoted by \bar{t} . The closure of an element $t \in \mathcal{T}_n^o$ is a knot iff $\Phi_n^o(t)$ is an n -cycle. The operation of closure defines a map from \mathcal{T}_n^o to \mathcal{L} the space of link types.

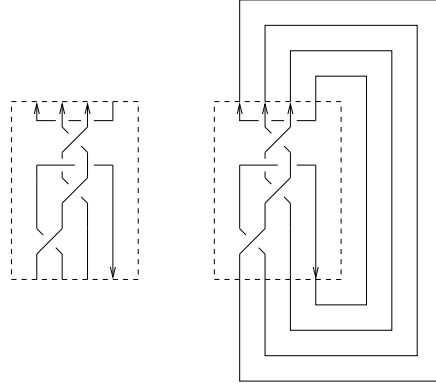


FIGURE 2. Left: An o -braid in \mathcal{B}_4^o . Right: Its closure. Here $o: I_4 \rightarrow \mathbb{Z}_2$ is given by: $o(1) = o(2) = o(3) = o(4) + 1 = 0 \in \mathbb{Z}_2$.

Now let $B_{n,e}^o = \Phi_n^{o-1}(e \in S^n)$ be the pure n th o -braid group. We have a map: $B_{n,e}^o \times \mathcal{T}_n^o \rightarrow \mathcal{T}_n^o$ which sends $(b, t) \in B_{n,e}^o \times \mathcal{T}_n^o$ to $bt \in \mathcal{T}_n^o$ by placing t on top of b as in the figure below. This map satisfies the condition $\Phi_n^o(bt) = \Phi_n^o(t)$.

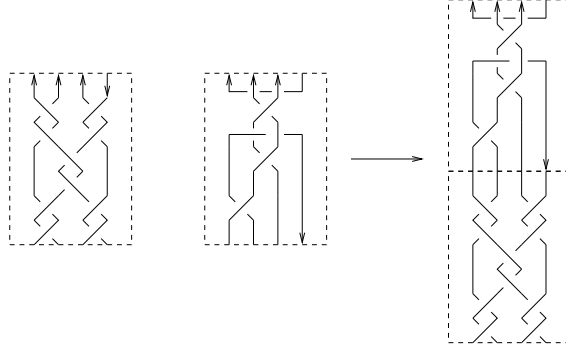


FIGURE 3. Left: A pure braid in \mathcal{B}_4^o . Center: A braid in \mathcal{T}_4^o . Right: Their composition.

1.2. Verginian moves. In this section we introduce terminology which on the one hand motivates the main theorem and on the other makes its statement concise. We formalize to a certain extent the notion of local move on knot diagrams.

Definition 1.1. A **Verginian move** or *operator*² on the space \mathcal{K} of all knots is a pair of tangles (T_1, T_2) with the same boundary acting on a knot K by scanning a projection of K for the appearance of T_1 (resp. T_2) and then replacing it by T_2 (resp. T_1). We say that K and J can be connected via μ moves if there is a finite sequence of applications of μ starting with a projection of K and ending with one of J . In this manner we get an equivalence relation \sim_μ on \mathcal{K} and on each equivalence class we get a metric (we denote them all by d_μ) defined as follows: If $K \sim_\mu J$ define $d_\mu(K, J)$ to be the minimum number of times we need to apply μ to pass from a projection of K to one of J . The number $|\mathcal{K}/\sim_\mu| - 1$ is called the **unknotting deficiency** of the move μ . A Verginian move is called an **unknotting operation** if $\mathcal{K}/\sim_{\mu u} = \{[O]\}$ i.e. if its unknotting deficiency is 0.

For every Verginian operator we get numerical knot invariants on every μ -equivalence class by fixing one base knot for every element of \mathcal{K}/\sim_μ and taking the distance of any other knot in the same μ -class to that base knot. The ordinary unknotting number uses as base knot the unknot. Any numerical invariant thus obtained is of non- finite type. The issue therefore arises of how (if at all) such numerical invariants are encoded into the finite type ones especially in view of the conjectured knot classification by finite type invariants.

Verginian moves can be composed as follows. Let μ_i , $i = 0, 1$ be two such moves. Denote their constituting coordinate tangles by $p_j(\mu_i)$ where p_j is the projection on the j th factor $j = 1, 2$. Then the union move $\mu_1 \cup \mu_2$ is that move which acts on knots by scanning knot projections for the appearance of $p_j(\mu_i)$ and replacing it by $p_{j+1}(\mu_i)$, where j is to be read modulo 2.

1.3. The Definition of the move and the statement of the Theorem. Now we are ready to define the move we want to consider and state the main Theorem

²These are called local moves in the literature and sometimes Gordian moves as R. Wendt coined them. However in the story involving Alexander the Great, Gordian is the knot and not the cutting method. So one should call such a move a Great Alexandrian move. Due however to the existence of a great Alexander in topology I chose to refer to them as Verginian operations after the birthplace of Alexander the Great.

of the paper. Let $C_{k,\vec{d},o}$ be the local move defined by $(BH_{\vec{d}}^o(k), e) \in \mathcal{B}_{k+2}^o \times B_{k+2}^o$, where

$$BH_{\vec{d}}^o(k) = \left(\prod_{i=1}^k \sigma_i^{d_i} \right) \sigma_{k+1}^{d_{k+1}} \left(\prod_{i=1}^k \sigma_{k+1-i}^{-d_{k+1}-i} \right) \left(\prod_{i=2}^k \sigma_i^{d_i} \right) \sigma_{k+1}^{-d_{k+1}} \left(\prod_{i=2}^k \sigma_{k+2-i}^{-d_{k+2}-i} \right),$$

with $d_i \in \{\pm 2\}$, $\vec{d} = (d_1, \dots, d_{k+1})$ and σ_i are the standard generators of B_{k+2}^o (here $o: I_{k+2} \rightarrow \mathbb{Z}_2$). This is an element of B_{k+2}^o and we will use the same symbol to denote the geometric tangle defined by it.

Remark 1.1. *The reader may check that the standard closure of $BH_{\vec{d}}^o(k)$ in nothing but the k th iterated Bing-double of the Hopf-link, for any \vec{d}, o and that therefore the union $\cup_{o,\vec{d}} C_{k,o,\vec{d}}$ is nothing but K . Habiro's C_{k+1} -move ([2]). In the language of Verginian moves:*

$$C_{k+1} = \bigcup_{o,\vec{d}} C_{k,o,\vec{d}}$$

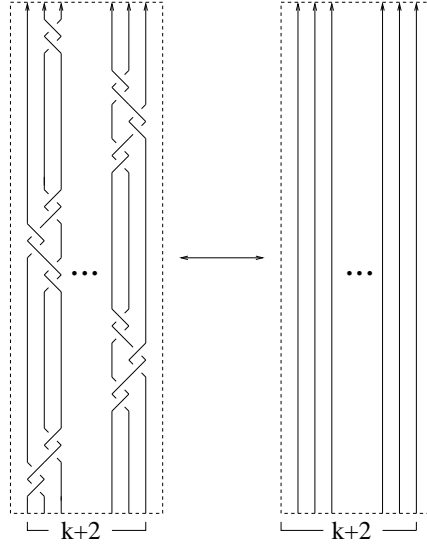


FIGURE 4. The move $C_{k,\vec{d},o}$ for $d_i = 2$ for all i and $\text{Im}(o) = 0$

We will write $d_{k,\vec{d},o}$ instead of $d_{C_{k,\vec{d},o}}$ for the metrics $C_{k,\vec{d},o}$ defines on each of its equivalence classes.

Now we are ready to state the theorem of this paper after we define some braids in B_{k+2}^o . Let $\vec{u} \in \mathbb{Z}_2^k$ be a vector parameter and denote its coordinates by u_i . For any such \vec{u} define $\vec{u}+1$ to be the vector $\vec{u} + (1, 1, \dots, 1)$. Define then $W_{\vec{u}} = a_1 \cdots a_k$ with $a_i = e$ or σ_i^2 according as u_i is 0 or 1. Define also $W_{\vec{u}}^r = a_k \cdots a_1$ with $a_i = e$ or σ_i^2 according as u_i is 0 or 1.

Theorem 1.1. *Suppose that K and J are two knots with $d_{k,\vec{d},o}(K, J) = 1$ for some \vec{d}, o, k . Then we can write K as the closure $\overline{BH_{\vec{d}}^o(k)T}$ of $BH_{\vec{d}}^o(k)T$ and J as the closure \overline{T} of T for some $T \in \mathcal{B}_{k+2}^o$. Furthermore:*

$$v_{k+1}(K) - v_{k+1}(J) = s(d_{k+1})(-1)^{o(k+1)o(k+2)} \sum_{\vec{u} \in Z_2^k} s_{\vec{u}, \vec{d}, o} v_{k+1}(\overline{W_{\vec{u}} \sigma_{k+1}^2 W_{\vec{u}+1}^r x})$$

for any Vassiliev invariant v_{k+1} of degree $k+1$ and any $x \in \mathcal{T}_{k+2}^o$ such that $\Phi_{k+2}^o(T) = \Phi_{k+2}^o(x)$. The sign $s_{\vec{u}, \vec{d}, o}$ is given by:

$$s_{\vec{u}, \vec{d}, o} = \prod_{i=1}^k (-1)^{u_i+1} s(d_i) (-1)^{o(i)o(i+1)}$$

Remark 1.2. For $k=1$ this is essentially the result in [4] from which this paper was originally inspired. During its conception the author was made aware of the newly published [3] where a result similar in philosophy is obtained. The result of this paper is different than that of [3] in terms of approach, degree of analysis and scope.

- Notice that because $d_{C_{k+1}}(K, J) = 1$ implies $d_{k, \vec{d}, o}(K, J) = 1$ for some \vec{d} and some o the theorem states that if $d_{C_{k+1}}(K, J) = 1$ then the formula of Theorem 1.1 holds for some \vec{d} and o and any $x \in \mathcal{T}_{k+2}^o$ such that $\Phi_{k+2}^o(x) = \Phi_{k+2}^o(T)$.
- Notice that it follows from the formula of Theorem 1.1 that, for fixed K, J, o, \vec{d} with $d_{k, \vec{d}, o}(K, J) = 1$, the degree $k+1$ invariants do not see the cycle $\Phi_{k+2}^o(x)$. In fact it is messy but not hard to convince oneself that the knots which appear on the right hand side of the formula of Theorem 1.1 when taken as a set with multiplicities remains invariant under the choice of x in the following sense. Let σ and σ' be two $k+2$ -cycles in S_{k+2} . Then there exist tangles x and x' with $\Phi_n^o(x) = \sigma$ and $\Phi_n^o(x') = \sigma'$ such that set of knots $\overline{W_{\vec{u}} \sigma_{k+1}^2 W_{\vec{u}+1} x}$ is equal to the set of knots $\overline{W_{\vec{u}} \sigma_{k+1}^2 W_{\vec{u}+1} x'}$.

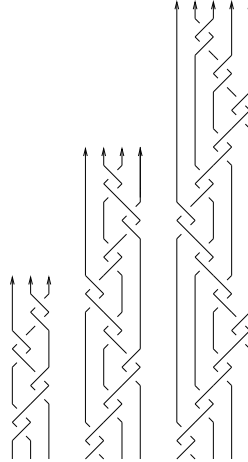


FIGURE 5. $BH_{\vec{d}}^o(k)$ for $k=1, 2, 3$, $\text{Im}(o) = 0$, $d_i = 2$ all i .

If two knot K and J differ by replacing one of the braids above with the corresponding trivial one then the difference of their degree k ($=2, 3, 4$) invariants is

given as in the figure below. We assume for the sake of simplicity and concreteness that T maps via Φ_{k+2} to the $k+2$ cycle $(1(k+2)(k+1)k \dots 2)$.

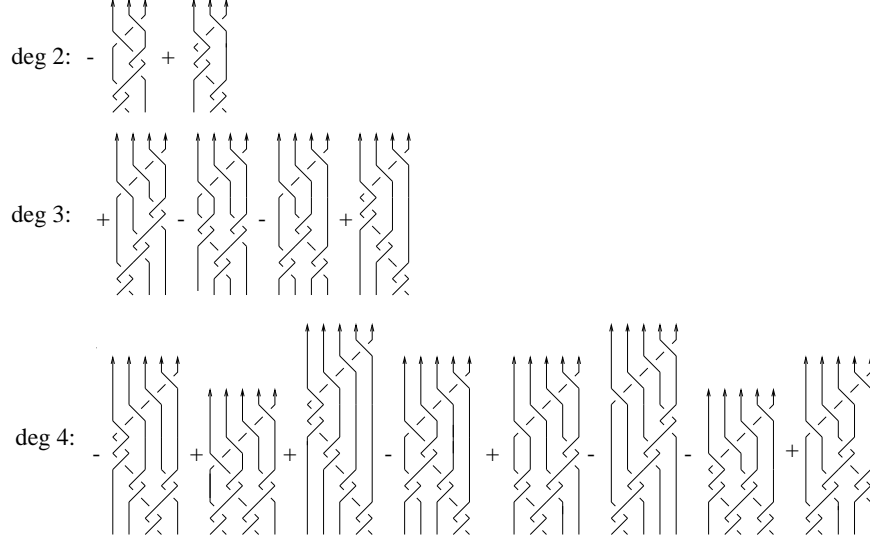


FIGURE 6. The difference of Vassiliev invariants. For every tangle get a knot by closing. In place of every knot the value of its respective invariant.

2. The proof

We introduce some notation and remind the reader of standard conventions. Once we are set with all of that the proof will be straightforward. We adopt the convention that when we write equations of Vassiliev invariants with several (singular) knots involved we draw (or write whichever the case maybe) only that part of the knot where the knots may differ. Also in an equation of Vassiliev invariants we can write a knot K instead of the value $v(K)$ of the invariant v on K if v is understood. We will be writing the generators of B_n^o as σ_i . There is an ambiguity of notation here but we will avoid it by declaring o every time we write σ_i 's. We will also consider singular o -braids. We will write σ_i^\times for the case where we have a self intersection at the corresponding place. In the figure below we can see an example of this notation.

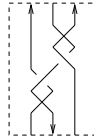


FIGURE 7. A 2-singular o -braid on 3 strings. Here $o: I_3 \rightarrow Z_2$ is given by $o(1) = o(3) = o(2) + 1 = 0 \in Z_2$. This singular braid can be written as: $\sigma_1^\times \sigma_1 \sigma_2^{-1} \sigma_2^\times$.

We will use the notation σ_i^+ instead of $\sigma_i \sigma_i^\times = \sigma_i^\times \sigma_i$ and σ_i^- instead of $\sigma_i^{-1} \sigma_i^\times = \sigma_i^\times \sigma_i^{-1}$ so that the example in the figure above can be written as $\sigma_1^+ \sigma_2^-$. For any non-zero real number x we will write $s(x)$ for $\frac{|x|}{x}$. Define $s_x = \pm$ whenever $s(x) = \pm 1$. Instead of s_{d_i} we will write s_i . We need this symbol in order to be able to express ambiguity of the type $\sigma_i^\pm = \sigma_i^\times \sigma_i^{\pm 1}$. We draw the readers attention to the difference between σ_i^\pm and $\sigma_i^{\pm 1}$. The first are singular words the second ones non-singular. Then if a knot projection contains $\sigma_i^{d_i} \in \mathcal{B}_n^o$ we can express the Birman-Lin condition ([1]) at a crossing of $\sigma_i^{d_i}$ by writing: $\sigma_i^{d_i} = e + s(d_i)(-1)^{o(i)+o(i+1)}\sigma^{s_i}$ or simply $\sigma_i^{d_i} = e + so(i)\sigma^{s_i}$ if we let $so(i) = s(d_i)(-1)^{o(i)+o(i+1)}$.

Now suppose that we are given a word in some B_n^o and we want to apply the Birman-Lin condition:



FIGURE 8. The Birman-Lin Condition.

Let's say that we have $\sigma_1^2 \sigma_2^2 \sigma_1^{-2} \sigma_2^{-2} \in \mathcal{B}_3$ and that we have marked a crossing where we will apply the Birman-Lin condition. We can then write out the tree of possibilities and at the end look at our expression or we can write the tree in the form of an expression like: $(e + \sigma_1^+)(e + \sigma_2^+)(e - \sigma_1^+)(e - \sigma_2^+)$ and then just multiply through. The reader may check for himself that such a product is meaningful as well as check the steps we give below.

$$\begin{aligned}
 (e + \sigma_1^+)(e + \sigma_2^+)(e - \sigma_1^-)(e - \sigma_2^-) &= \\
 (e + \sigma_1^+)e(e - \sigma_1^-)(e - \sigma_2^-) + (e + \sigma_1^+)\sigma_2^+(e - \sigma_1^-)(e - \sigma_2^-) &= \\
 (e + \sigma_1^+)(e - \sigma_1^-)(e - \sigma_2^-) + (e + \sigma_1^+)\sigma_2^+(e - \sigma_1^-)(e - \sigma_2^-) &= \\
 (e + \sigma_1^+ - \sigma_1^- - \sigma_1^+ \sigma_1^-)(e - \sigma_2^-) + (e + \sigma_1^+)\sigma_2^+(e - \sigma_1^-)(e - \sigma_2^-) &=
 \end{aligned} \tag{1}$$

but now notice that $\sigma_1^+ \sigma_1^- = \sigma_1^\times \sigma_1^\times = \sigma_1^+ - \sigma_1^-$ so we can continue as follows:

$$\begin{aligned}
 (e - \sigma_2^-) + (e + \sigma_1^+)\sigma_2^+(e - \sigma_1^-)(e - \sigma_2^-) &= \\
 e - \sigma_2^- + \sigma_2^+(e - \sigma_2^-) + \sigma_1^+\sigma_2^+(e - \sigma_2^-) &= \\
 -\sigma_2^+\sigma_1^-(e - \sigma_2^-) - \sigma_1^+\sigma_2^+\sigma_1^-(e - \sigma_2^-) &= \\
 e - \sigma_2^- + \sigma_2^+ - \sigma_2^+\sigma_2^- + \sigma_1^+\sigma_2^+ - \sigma_1^+\sigma_2^+\sigma_2^- &= \\
 -\sigma_2^+\sigma_1^- + \sigma_2^+\sigma_1^-\sigma_2^- - \sigma_1^+\sigma_2^+\sigma_1^- + \sigma_1^+\sigma_2^+\sigma_1^-\sigma_2^- &= \\
 e + \sigma_1^+\sigma_2^+ - \sigma_2^+\sigma_1^- + O_3 &
 \end{aligned} \tag{2}$$

where O_3 contains singular braids with at least three singularities. So if this was a calculation of v_2 then O_3 would vanish and hence we would have Okada's result in [4]. Examining this calculation the reader may get familiar with this notation quicker than it would take to write out all the details. The idea is that if we have a knot projection containing a braid and we intend to apply the Birman-Lin condition only locally on the braid in order to express its Vassiliev invariant of some degree in terms of singular knots resulting from these considerations and we therefore adopt the convention that we only write the braid instead of the invariant then we can perform the calculations by treating singular braids as objects in an algebra of sorts.

We now proceed to proving Theorem 1.1 in its complete generality.

Suppose that K and J are two knots such that $d_{k,\vec{d},o}(K, J) = 1$ for some \vec{d}, o . Then we can write K as $\overline{BH_d^o(k)T}$ the closure of $BH_d^o(k)T$ and J as \overline{T} for some $T \in \mathcal{T}_{k+2}^o$.

We now compute the degree $k+1$ Vassiliev invariant of K . Because we will only use the Birman-Lin Condition on $BH_d^o(k)$ we can write $v_{k+1}(BH_d^o(k))$ instead of $v_{k+1}(K)$ and in fact since we have declared that we are computing v_{k+1} we can drop the v_{k+1} all together and write $BH_d^o(k)$ instead of $v_{k+1}(BH_d^o(k))$.

$$\begin{aligned} BH_d^o(k) = & \prod_{i=1}^k (e + so(i)\sigma_i^{s_i})(e + so(k+1)\sigma_{k+1}^{s_{k+1}}) \prod_{i=1}^k (e - so(k+1-i)\sigma_{k+1-i}^{-s_{k+1-i}}) \\ & \prod_{i=2}^k (e + so(i)\sigma_i^{s_i})(e - so(k+1)\sigma_{k+1}^{-s_{k+1}}) \prod_{i=2}^k (e - so(k+2-i)\sigma_{k+2-i}^{-s_{k+2-i}}) \end{aligned} \quad (3)$$

We will expand this expression in two ways. Once starting from $(e + so(k+1)\sigma_{k+1}^{s_{k+1}})$ and proceeding radially in both directions and once from $(e \pm so(1)\sigma_1^{\pm s_1})$. Comparing the two expressions will give us the desired result. The key relation is this: $\sigma_i^+ \sigma_i^- = \sigma_i^\times \sigma_i^\times = \sigma_i^+ - \sigma_i^-$ by means of which we have:

$$(e + so(i)\sigma_i^{s_i})(e - so(i)\sigma_i^{-s_i}) = e + so(i)\sigma_i^{s_i} - so(i)\sigma_i^{-s_i} - \sigma_i^{s_i}\sigma_i^{-s_i} = e. \quad (4)$$

We begin by expanding radially from the peak $(e + so(k+1)\sigma_{k+1}^{s_{k+1}})$ down the hills on both sides. Before we begin we outline the plan. We will expand with an eye on separating the final expression into sums of singular words which contain σ_1^\pm and ones which do not. We will now argue that $BH_d^o(k)$ is given by:

$$BH_d^o(k) = e + \sum_{\vec{u} \in Z_2^k} so_{\vec{u}} U_{\vec{u}} \sigma_{k+1}^{s_{k+1}} U_{\vec{u}+1}^{-1} + O \quad (5)$$

where $W_{\vec{u}} = b_1 \dots b_n$ with $b_i = e$ or $\sigma_i^{s_i}$ depending on whether the i th coordinate of \vec{u} is 0 or 1. $W_{\vec{u}}^{-1}$ is the inverse of the singular word $W_{\vec{u}}$ (upside down crossing reversed singularities follow accordingly). O is a sum of singular words which do not contain σ_1^\pm . The sign $so_{\vec{u}}$ is given by:

$$so_{\vec{u}} = \prod_{i=1}^k so(i)(-1)^{u_i+1}$$

To see that equation (5) holds is not hard. It falls basically out of the relations of equation (4) as follows.

Let

$$A = \prod_{i=1}^k (e + so(i)\sigma_i^{s_i})(e + so(k+1)\sigma_{k+1}^{s_{k+1}}) \prod_{i=1}^k (e - so(k+1-i)\sigma_{k+1-i}^{-s_{k+1-i}}),$$

and begin expanding it from the middle outward. Then using equation (4) we get:

$$A = e + so(k+1) \prod_{i=1}^k (e + so(i)\sigma_i^{s_i}) \sigma_{k+1}^{s_{k+1}} \prod_{i=1}^k (e - so(k+1-i)\sigma_{k+1-i}^{-s_{k+1-i}})$$

Now notice that:

$$\begin{aligned}
& so(k+1) \prod_{i=1}^k (e + so(i)\sigma_i^{s_i}) \sigma_{k+1}^{s_{k+1}} \prod_{i=1}^k (e - so(k+1-i)\sigma_{k+1-i}^{-s_{k+1-i}}) = \\
& so(k+1)\sigma_{k+1}^{s_{k+1}} + \\
& so(k)so(k+1) \prod_{i=1}^{k-1} (e + so(i)\sigma_i^{s_i}) \sigma_k^{s_k} \sigma_{k+1}^{s_{k+1}} \prod_{i=2}^k (e - so(k+1-i)\sigma_{k+1-i}^{-s_{k+1-i}}) \\
& - so(k)so(k+1) \prod_{i=1}^{k-1} (e + so(i)\sigma_i^{s_i}) \sigma_{k+1}^{s_{k+1}} \sigma_k^{-s_k} \prod_{i=2}^k (e - so(k+1-i)\sigma_{k+1-i}^{-s_{k+1-i}}) \\
& - so(k+1) \prod_{i=1}^{k-1} (e + so(i)\sigma_i^{s_i}) \sigma_k^{s_k} \sigma_{k+1}^{s_{k+1}} \sigma_k^{-s_k} \prod_{i=2}^k (e - so(k+1-i)\sigma_{k+1-i}^{-s_{k+1-i}})
\end{aligned} \tag{6}$$

Inspecting equation (6) we can get equation (5) as follows. The middle two summands are the only ones which have a chance of leading to singular words containing σ_1^\pm . That the first summand does not it is clear because it does not contain σ_1^\pm and cannot be expanded further. The last summand will not lead to singular words containing σ_1^\pm because the only way it has a chance to do this is through words where for each i at least one of $\sigma_i^{\pm s_i}$ remains left or right of $\sigma_{k+1}^{s_{k+1}}$. This is because as soon as for some i this does not happen the σ_1^\pm 's will disappear using equation (4). But if this is the case since we are computing invariants of degree $k+1$ the terms which retain the chance of containing $\sigma_i^{\pm s_i}$ will disappear before the expansion reaches σ_1^\pm .

Now we begin expanding the expression for $BH_d^o(k)$ in the second way. Write $BH_d^o(k) = (e + so(1)\sigma_1^{s_1})M(e - so(1)\sigma_1^{-s_1})N$ and expand to get:

$$BH_d^o(k) = MN + so(1)\sigma_1^{s_1}MN - so(1)M\sigma_1^{-s_1}N - \sigma_1^{s_1}M\sigma_1^{-s_1}N \tag{7}$$

Notice that using equation (4) $MN = e$. Also all the other terms contain σ_1^\pm . Comparing equations (5) and (7) we conclude that O is zero and that

$$BH_d^o(k) = e + \sum_{\vec{u} \in Z_2^k} so_{\vec{u}} U_{\vec{u}} \sigma_{k+1}^{s_{k+1}} U_{\vec{u}+1}^{-1}. \tag{8}$$

But now remember that (8) is an equation of degree $k+1$ Vassiliev invariants of singular knots which are identical away from the singular words appearing in the equation. On the other hand in the equation every singular knot has $k+1$ singularities and hence it's crossings can be changed at will. This means that we can for example for each of this singular braids make any choice of a tangle $D_{\vec{u}}$ with which to close them as long as $\Phi_n^o(D_{\vec{u}}) = \Phi_n^o(B)$. Also we can change the crossings of the singular braids so that instead of $U_{\vec{u}+1}^{-1}$ we have $U_{\vec{u}+1}^r = a_n \dots a_1$ with $a_i = e$ or σ_i^+ according as $u_i = 1$ or 0 respectively. Also we can assume that $U_{\vec{u}}$ is given by $a_1 \dots a_n$ with $a_i = e$ or σ_i^+ according as $u_i = 0$ or 1 respectively So we can write equation (8) as:

$$BH_d^o(k) = e + \sum_{\vec{u} \in Z_2^k} so_{\vec{u}} \overline{U_{\vec{u}} \sigma_{k+1}^{s_{k+1}} U_{\vec{u}+1}^r D_{\vec{u}}}, \tag{9}$$

for any choice of $D_{\vec{u}}$ as above. If the choice is made so that $D_{\vec{u}} = x = \text{const.}$ for all \vec{u} then we can desingularize via the Birman-Lin Condition and find that all but the desired terms are canceled because they pair up with opposite signs on the level of braids already. \square

3. Closing Remarks

We would like now to see what Theorem 1.1 says about the Vassiliev invariants of degree $k+1$ of C_{k+1} -equivalent knots i.e. (via Habiro's Theorem) of knots all of whose degree k invariants agree. Consider the o -braid group $\mathcal{B}_{n(k+2)}^o$ for $o: I_{n(k+2)} \rightarrow Z_2$ an orientation function, the o -tangle set $\mathcal{T}_{n(k+2)}^o$ and the function $\Phi_{n(k+2)}^o: \mathcal{T}_{n(k+2)}^o \rightarrow S_{n(k+2)}$. Consider also the pure braid group $\mathcal{B}_{n(k+2),e}^o = \Phi_{n(k+2)}^o{}^{-1}(e \in S_{n(k+2)})$.

In $\mathcal{B}_{n(k+2),e}^o = \Phi_{n(k+2)}^o{}^{-1}(e \in S_{n(k+2)})$ consider the words as in the figure below:

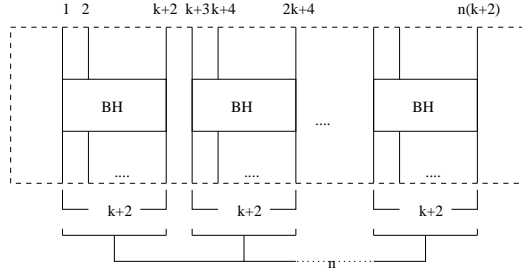


FIGURE 9. The words $BH_d^o(n, k) \in \mathcal{B}_{n(k+2),e}^o$ formed by sticking sideways the words $BH_{d_j}^{o_j}(k)$ (inside each box labeled BH) defined on the points $I_{j,k+2} = \{j+1, \dots, j+k+2\}$ for $j = 0, \dots, n-1$. The orientation functions o_j are given by restricting o on $I_{j,k+2}$.

Form also words $W_{\vec{u}_j} \in \mathcal{B}_{n(k+2),e}^o$ as in the figure below:

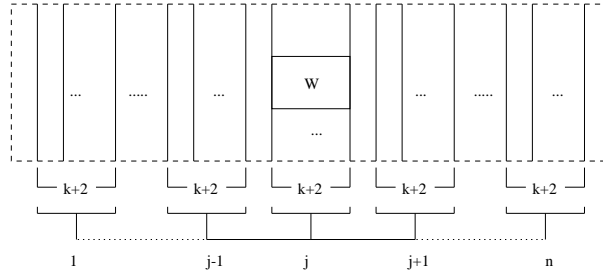


FIGURE 10. The words $W_{\vec{u}_j} \in \mathcal{B}_{n(k+2),e}^o$ formed by means of the word $W_{\vec{u}}$ (inside the box labeled W) defined on the points $I_{j,k+2} = \{j+1, \dots, j+k+2\}$ for $j = 0, \dots, n-1$.

Define now the (Habiro)-set of pure o -braids $\mathcal{B}_{n(k+2),H}^o$ containing all braids of the form $BH_d^o(n, k)$. This is nothing but the preimage under the closure map (let

us call it κ here), $\kappa: \mathcal{T}_{n(k+2)}^o \rightarrow \mathcal{L}$, where \mathcal{L} is the space of all links, of the k th iterated Bing-double of the Hopf link. Then Habiro's Theorem that "two knots are C_{k+1} -equivalent iff all their degree k Vassiliev invariants coincide" can be restated as saying that two knots are C_{k+1} -equivalent iff there is a $T_{n_k(k+2)} \in \mathcal{T}_{n_k(k+2)}^o$ with $\Phi_{n_k(k+2)}^o(T_{n_k(k+2)})$ a $n_k(k+2)$ -cycle $\in S_{n_k(k+2)}$ for some n_k and o such that while $J = \overline{T_{n_k(k+2)}}$, K is in the image of the composite map below (where the first map is the composition defined in Figure 3)

$$\mathcal{B}_{n_k(k+2)}^o \times \{T_{n_k(k+2)}\} \rightarrow \mathcal{T}_{n_k(k+2)}^o \xrightarrow{\kappa} \mathcal{K}$$

Now let

$$\mathcal{B}_{n_k(k+2),V}^o = \bigcup_{j, \vec{u}_j} \{W_{\vec{u}_j} \sigma_{j(k+2)+k+1} W_{\vec{u}_j}^r\}$$

be the (Vassiliev)-set of braids. Consider the map:

$$\mathcal{B}_{n_k(k+2),V}^o \times \Phi_{n_k(k+2)}^{o-1}(T_{n_k(k+2)}) \rightarrow \mathcal{T}_{n_k(k+2)}^o \xrightarrow{\kappa} \mathcal{K}$$

and consider the sets of knots \mathcal{K}_x obtained via this map for any choice of $x \in \Phi_{n_k(k+2)}^{o-1}(T_{n_k(k+2)})$.

We can now say that if K and J are C_{k+1} -equivalent then:

$$v_{k+1}(K) - v_{k+1}(J) = \sum_{K \in \mathcal{K}_x} s(K) v_{k+1}(K),$$

for any choice of x , where $s(K)$ is a sign; or we can write precisely:

$$v_{k+1}(K) - v_{k+1}(J) = \sum_{j=1}^{n_k-1} \sum_{\vec{u}_j \in Z_2^k} so_{\vec{u}_j} v_{k+1}(\overline{W_{\vec{u}_j} \sigma_{j(k+2)+k+1}^2 W_{\vec{u}_j}^r x}), \quad (10)$$

and this for any $x \in \mathcal{T}_{n_k(k+2)}^o$ which satisfies $\Phi_{n_k(k+2)}^o(x) = \Phi_{n_k(k+2)}^o(T_{n_k})$, where $W_{\vec{u}_j} = a_{j(k+2)+1} \dots a_{j(k+2)+k}$ with $a_{j(k+2)+1} = e$ or $\sigma_{j(k+2)+1}^2$ according as the i th coordinate u_{ji} of \vec{u}_j is equal to 0 or 1 and $W_{\vec{u}_j}^r = a_{j(k+2)+k} \dots a_{j(k+2)+1}$. The d_{ji} 's take values in $\{\pm 2\}$. The signs $so_{\vec{u}_j}$ are given by:

$$so_{\vec{u}} = \prod_{i=1}^k so(i) (-1)^{u_{ji}+1}$$

Here is an example of the general statement. Suppose that the C_2 distance of two knots K and J is equal to 2. Then they may differ by a braid which looks like:

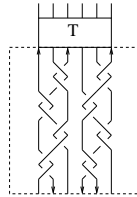


FIGURE 11. Two knots whose C_2 distance is equal to 2 may differ by such a tangle or its variants with respect to choices of d_i 's and o 's (string orientations).

Then their degree 2 Vassiliev Invariants differ by:

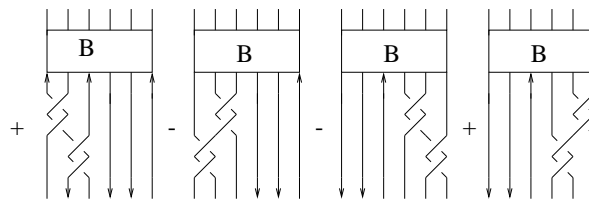


FIGURE 12. The difference of degree 2 Vassiliev invariants. B is any o -braid which induces the same full cycle in S_6 as T . Get knots by closing.

A couple of final remarks:

- One of the consequences of Theorem 1.1 is that the difference of the normalized degree $k + 1$ Vassiliev invariants of knots with $d_{C_{k+1}}$ -distance equal to one is bounded. More generally if we take the $d_{C_{k+1}}$ -ball of radius r then the difference of the normalized degree $k + 1$ Vassiliev invariants of any two knots inside this ball is bounded by a constant which depends only on r .
- It appears as though the permutation defined by $T \in \mathcal{T}_{k+2}^o$ (the image under Φ_{k+2}^o) may define a new invariant of knots which is not predicted by Vassiliev invariants.
- It seems as though the cardinality of the set of knots which appear in the formula of equation (10) should provide a (probably sharp) bound on the number of linearly independent Vassiliev invariants of degree $k + 1$. There appear to be many duplicates in their braid representation which when moded out should lead to such a bound (see Remark 1.2). We should also mention that it follows from the proof of Theorem 1.1 that the formula of the Theorem and its generalization hold also iff in the statement we replace all σ_i^2 's by σ_i^+ 's.

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